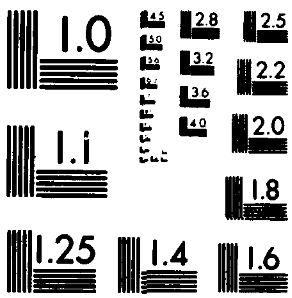
AD-A110 478 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
ERROR ESTIMATES FOR GAUSSIAN QUADRATURE AND WEIGHTED-L1 POLYNOM--ETC(U)
AUG 81 R A DEVORE, L R SCOTT DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2269 NL

END
DATE FILMED
3 82
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL

ADA110478

MRC Technical Summary Report #2269

ERROR ESTIMATES FOR GAUSSIAN QUADRATURE AND WEIGHTED-L^1 POLYNOMIAL APPROXIMATION

Ronald A. DeVore and L. Ridgway Scott

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

August 1981

Received July 2, 1981

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ERROR ESTIMATES FOR GAUSSIAN QUADRATURE AND WEIGHTED-L^1 POLYNOMIAL APPROXIMATION

Ronald A. DeVore and L. Ridgway Scott

Technical Summary Report #2269
August 1981

ABSTRACT

Error estimates for Gaussian quadrature are given in terms of the number of quadrature points and smoothness properties of the function whose integral is being approximated. An intermediate step involves a weighted-L^1 polynomial approximation problem which is treated in a more general context than that specifically required to estimate the Gaussian quadrature error.

AMS(MOS) Subject classification: 65D30, 41A55, 41A10

Key Words: Gaussian quadrature, polynomial approximation

Work Unit No. 3 - Numerical Analysis and Computer Science

Sponsored by the United States Army under Grant No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS-81-01661 and MCS-81-01631.

SIGNIFICANCE AND EXPLANATION

This paper gives error estimates for Gaussian quadrature, one of the most widely used methods for approximating integrals. It also studies a related approximation problem that has more general potential applications to the study of numerical approximations. A particular anticipated use of the results on Gaussian quadrature is for error estimates of the discrete ordinates method for the transport equation, by the second author in collaboration with Professor J. Pitkäranta of the University of Helsinki. The transport equation appears as a model equation in nuclear engineering, astrophysics and climatology, to name only a few instances.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ERROR ESTIMATES FOR GAUSSIAN QUADRATURE AND WEIGHTED-L^1 POLYNOMIAL APPROXIMATION

Ronald A. DeVore and L. Ridgway Scott

1. Introduction

The purpose of this note is to establish error estimates for Gaussian quadrature that reflect the fact that the accuracy is not degraded by certain singularities of the function to be integrated at the boundaries of the interval of integration. While it is well known that polynomial approximation can achieve greater accuracy at the boundary than in the interior (cf. Timan [3], p. 262), error estimates for Gaussian quadrature of the explicit type given here have apparantly not been presented before. The basic thrust of these estimates is to show that the error in N-point Gaussian quadrature for approximately integrating f on $[-1,1]$ is bounded by

$$C_s N^{-s} \int_{-1}^{1} |f^{(s)}(x)|(1 - x^2)^{s/2} dx \tag{1.1}$$

for all integers $s \leq 2N$ such that above integral makes sense. Here C_s is a constant independent of N and f.

Our technique of proof is in two parts. Firstly, we analyze a Peano kernel for the quadrature error. This has the effect of establishing the bound (1.1) for $s = 1$ and reducing further estimates to a weighted-L^1 approximation problem for polynomials. This part of the analysis is presented in section 2. Secondly, we consider weighted-L^1 approximation by polynomials and prove estimates for the error. This is done in section 3.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS-81-01661 and MCS-81-01631.

Finally, the estimates of sections 2 and 3 are combined in section 4 to give error bounds for Guassian quadrature of the type (1.1) for all $s \geq 1$. The calculations in section 2 are done specifically for Gaussian quadrature, although they clearly extend to more general quadrature approximations based on orthogonal polynomials for weighted integrals.

The weighted-L^1 approximation estimates show that, for a rather general class of weights $w > 0$ and sufficiently smooth f, there is a polynomial P of degree at most N so that

$$(1.2) \qquad \int_{-1}^{1} |f(x) - P(x)| w(x)\, dx \leq C_s N^{-s} \int_{-1}^{1} |f^{(s)}(x)| (1-x)^{s/2} w(x)\, dx$$

for any s such that $N+1 \geq s \geq 1$. In the case $w \equiv 1$ and $s = 1$, (1.2) has been established by N. X. Ky [1].

2. Error estimates for Gaussian quadrature

Consider Gaussian quadrature approximation of the form

$$\int_{-1}^{1} f(x)\,dx \sim \sum_{j=1}^{N} \omega_j\, f(x_j) =: I_N(f) \tag{2.1}$$

($\{x_j\}$ are the zeros of the Legendre polynomials and $\{\omega_j\}$ are the integrals of the associated Lagrange interpolation polynomials, cf. G. Szegö [2]). The ordering $-1 < x_1 < \cdots < x_N < 1$ will be assumed, and we introduce $x_0 :=$ -1 and $x_{N+1} := 1$. We wish to establish estimates of the error

$$e_N(f) := \int_{-1}^{1} f(x)\,dx - I_N(f) \tag{2.2}$$

in terms of N and properties of f. For example, the Peano kernel theorem allows us to write

$$e_N(f) = \int_{-1}^{1} K(t) f'(t)\,dt,$$

at least for smooth f, where $K(t) = e_N(H_t)$, $|t| \le 1$, (cf. (3.3) below) and H_t is the Heaviside function

$$H_t(x) := \begin{cases} 0 & x < t \\ 1 & x \geq t. \end{cases} \tag{2.3}$$

It follows that

$$K(t) = 1 - t - \sum_{x_j > t} \omega_j = \sum_{x_j < t} \omega_j - t - 1.$$

The Chebyshev-Markov-Stieltjes inequality (cf. G. Szegö [2], p. 50) implies that

$$1 + x_j \le \sum_{i=1}^{j} \omega_i \le 1 + x_{j+1}, \quad j = 1,\cdots,N.$$

Therefore, for $j = 1,\cdots,N$,

$$x_{j-1} - x_j \le K(x_j-) \le 0 \le K(x_j+) \le x_{j+1} - x_j.$$

Let $\bar{K}$ denote the continuous, piecewise-linear function (with respect to the knots $x_1,\cdots,x_N$) which interpolates 0 at ± 1 and the value

$$d_j := \max\{x_j - x_{j-1}, x_{j+1} - x_j\} \quad \text{at} \quad x_j, \quad j = 1,\cdots,N.$$

Since K is itself piecewise linear,

$$|K(x)| \leq \bar{K}(x) \qquad \text{for all} \quad x \in [-1,1].$$

To bound the values d_j, recall (cf. G. Szegö [2], p. 122) that

$$x_j = -\cos\theta_j \quad \text{where} \quad (2N+1)\theta_j/\pi \in [2j-1,2j], \quad j = 1,\cdots,N. \tag{2.4}$$

Thus

$$\begin{aligned} x_j - x_{j-1} &= \cos\theta_{j-1} - \cos\theta_j \\ &= \int_{\theta_{j-1}}^{\theta_j} \sin\theta \, d\theta \\ &\leq (\theta_j - \theta_{j-1}) \max\{\sin\theta_{j-1}, \sin\theta_j\} \\ &\leq (3\pi/2N) \max\{\sin\theta_{j-1}, \sin\theta_j\}. \end{aligned} \tag{2.5}$$

Since the function $\theta \to \sin\theta/\theta$ is decreasing on $[0,\pi]$, (2.4) implies that

$$\tfrac{1}{4}\sin\theta_{j-1} \leq \sin\theta_j \leq 4\sin\theta_{j-1}, \quad 2 \leq j \leq N.$$

Thus

$$d_j \leq 6\pi \sin\theta_j/N = 6\pi\sqrt{1-\cos^2\theta_j}\,/\,N = 6\pi\sqrt{1-x_j^2}\,/N.$$

Since the function $x \to \sqrt{1-x^2}$ is concave, this implies that

$$|K(x)| \leq \bar{K}(x) \leq 6\pi\sqrt{1-x^2}\,/N \qquad \text{for} \quad x \in [-1,1]. \tag{2.6}$$

Thus we have proved that

$$|e_N(f)| \leq 6\pi N^{-1} \int_{-1}^{1} |f'(x)|\sqrt{1-x^2}\, dx.$$

Since (2.4) implies that

$$1/N \leq \sin \pi/(2N+1) \leq \sqrt{1-x^2}$$

for $x \in [x_1,x_N]$ and $N \geq 2$, (2.6) implies further that

$$|K(x)| \leq 6\pi(1-x^2) \tag{2.7}$$

for $x \in [x_1,x_N]$. But for $x \in [-1,x_1] \cup [x_N,1]$, $|K(x)| = 1 - |x|$. Thus (2.7) holds for all $x \in [-1,1]$, giving the following estimate:

$$|e_N(f)| \leq 6\pi \int_{-1}^{1} |f'(x)| \min\{\sqrt{1-x^2}\,/N,\ 1-x^2\}\, dx.$$

These estimates hold for all $f \in L^1([-1,1])$ whose weak derivative is integrable with respect to the weight $1 - x^2$ as can be seen by approximating f via smooth functions (cf. the definition of the space Y_w^s given later in section 3). Summarizing the above, we have the following:

Theorem 1. Let $e_N(f)$ denote the error in N-point Guassian quadrature applied to $f \in L^1([-1,1])$ (see (2.1-2) for definitions). If the weak derivative, f', of f is integrable with respect to the weight $1 - x^2$, then

$$|e_N(f)| \leq 6\pi \int_{-1}^{1} |f'(x)| \min\{\sqrt{1-x^2}/N,\ 1-x^2\}dx.$$

If f' is integrable with respect to $\sqrt{1-x^2}$, then

$$|e_N(f)| \leq \frac{6\pi}{N} \int_{-1}^{1} |f'(x)| \sqrt{1-x^2}\, dx.$$

Estimates involving higher derivatives of f can also be derived by estimating Peano kernels. For example, one may write, for any $1 \leq k \leq 2N$,

$$e_N(f) = \int_{-1}^{1} K_k(x) f^{(k)}(x)dx$$

($K_1 = K$ in the previous notation). K_k is a $C^{(k-2)}$ piecewise k-th degree polynomial (with knots $\{x_i\}$) satisfying $0 = K_k(\pm 1) = \cdots = K_k^{(k-1)}(\pm 1)$ and $K_k^{(k)} = (-1)^k$ between knots. Moreover, $K_k^{(k-1)}(x_j+) - K_k^{(k-1)}(x_j-) = \omega_j$ for $j = 1,\cdots,N$. Using these facts together with a special oscillation property of K_2, it can be shown that

$$|K_2(x)| \leq C \min\{(1-x^2)/N^2, (1-x^2)^2\}.$$

However, it becomes increasingly difficult to estimate the higher degree kernels K_k, $k \geq 3$. Thus the following approach proves more fruitful.

Observe that $e_N(f) = e_N(f - P)$ for any $P \in \mathcal{P}_{2N-1}$, where $\mathcal{P}_r$ denotes the set of polynomials of degree not exceeding r. Therefore, Theorem 1 implies that

$$(2.8) \qquad |e_N(f)| \leq \inf_{P \in \mathcal{P}_{2N-2}} \int_{-1}^{1} |f'-P|(x)\ w(x)dx,$$

where w is the weight function

$$w(x) = 6\pi \min\{\sqrt{1-x^2}/N,\ 1-x^2\}.$$

Thus, estimates of $e_N(f)$ are reduced to a weighted-L^1 approximation problem for polynomials. Such problems will be considered in the next section. The results of section 3 on approximation will be combined with the results of the present section in section 4 to give higher order estimates of the form

$$(2.9) \qquad |e_N(f)| \le C_s \int_{-1}^{1} |f^{(s)}(x)| \min\{(\sqrt{1-x^2}/N)^s,\ (1-x^2)^s\}dx$$

for arbitrary positive integers $s \le 2N$.

3. Weighted-L^1 approximation

In this section, we shall prove a weighted-L^1 polynomial approximation result of the form

$$\text{(3.1)}\quad \inf_{P\in \mathcal{P}_N} \int_{-1}^{1} |u-P|(x)w(x)\,dx \le C_{s,w} N^{-s}\int_{-1}^{1} |u^{(s)}(x)|w(x)(1-x^2)^{s/2}dx,$$

where $\mathcal{P}_N$ denotes the set of polynomials of degree not exceeding N, s is a positive integer, $C_{s,w}$ is a constant independent of N and u, and w is a positive, integrable weight-function to be discussed in more detail subsequently. Rather than state our abstract conditions on w and u initially, we shall develop them in the course of deriving (3.1). However, suffice it to say that (3.1) will be proved for a class of weights including the Jacobi weights

$$\text{(3.2)}\qquad w(x) = (1+x)^{\alpha}(1-x)^{\beta},\ \alpha,\beta > -1.$$

As a first step, recall the Heaviside function $H_t(x)$ defined in (2.3). If u is sufficiently smooth, we may write

$$\text{(3.3)}\qquad u(x) = P_1(x) + \frac{1}{(s-1)!}\int_{-1}^{1} H_t(x)(x-t)^{s-1}u^{(s)}(t)\,dt$$

for all $x \in [-1,1]$, where $P_1 \in \mathcal{P}_{s-1}$. Let $\lambda_t \in \mathcal{P}_{N-s+1}$ be an arbitrary family depending, say, piecewise continuously on t, and define

$$\text{(3.4)}\qquad P(x) = P_1(x) + \frac{1}{(s-1)!}\int_{-1}^{1} \lambda_t(x)(x-t)^{s-1}u^{(s)}(t)\,dt.$$

Then $P \in \mathcal{P}_N$, and Hölder's inequality and Fubini's theorem imply that

$$\text{(3.5)}\qquad \begin{aligned}&\int_{-1}^{1} |u-P|(x)w(x)\,dx\\ &\le \frac{1}{(s-1)!}\int_{-1}^{1} \{\int_{-1}^{1} |\lambda_t-H_t|(x)|x-t|^{s-1}w(x)\,dx\}|u^{(s)}(t)|\,dt.\end{aligned}$$

Thus to prove (3.1), it suffices to construct λ_t in such a way that

$$\text{(3.6)}\qquad \int_{-1}^{1} |\lambda_t-H_t|(x)|x-t|^{s-1}w(x)\,dx \le C_s N^{-s}(1-t^2)^{s/2}w(t).$$

For t near ± 1, this is relatively easy to do under the following assumption:

(A1) There is a constant A_1 *such that*

i) $$\int_t^1 w(x)\,dx < A_1(1-t)w(t) \quad \textit{for} \quad 0 < t < 1$$

and ii) $$\int_{-1}^t w(x)\,dx < A_1(1+t)w(t) \quad \textit{for} \quad -1 < t < 0.$$

Example 1. Assumption (A1) holds for the Jacobi weight $w(x) = (1+x)^{\alpha}(1-x)^{\beta}$ *provided* $\alpha,\beta > -1$.

Proof. To see this, it suffices to verify, say, i). Then

$$\int_t^1 w(x)\,dx = \int_t^1 (1+x)^{\alpha}(1-x)^{\beta}dx$$

$$< 2^{\max\{\alpha,0\}} \int_t^1 (1-x)^{\beta}dx$$

$$< 2^{\max\{\alpha,0\}}(1-t)^{1+\beta}/(1+\beta)$$

$$< [2^{|\alpha|}/(1+\beta)](1+t)^{\alpha}(1-t)^{1+\beta}$$

$$= A_1(1-t)w(t). \quad //$$

Lemma. Suppose assumption (A1) holds. For $i = 0,1$, *let* λ^i *denote the functions given by*

$$\lambda^i(x) = i \quad \textit{for all } x \in [-1,1].$$

Then choosing $\lambda_t = \lambda^0$ *for* $t > 0$ *and* $\lambda_t = \lambda^1$ *for* $t < 0$ *gives*

$$\int_{-1}^1 |\lambda_t - H_t|(x)|x-t|^{s-1}w(x)\,dx < A_1(1-t^2)^s w(t)$$

for all $t \in [-1,1]$.

Proof. Suppose $t > 0$. From (2.3) and (A1), we have

$$\int_{-1}^{1} |\lambda_t - H_t|(x)|x-t|^{s-1}w(x)\,dx = \int_t^1 |x-t|^{s-1}w(x)\,dx$$

$$\leq (1-t)^{s-1}\int_t^1 w(x)\,dx$$

$$\leq A_1(1-t)^s w(t)$$

$$\leq A_1(1-t^2)^s w(t).$$

The case $t \leq 0$ is similar. //

Corollary. Let $\kappa \geq 1$ be arbitrary, and suppose (A1) holds. Then for any $N \geq 1$ and $t \in [-1,1]$ such that $\sqrt{1-t^2} \leq \kappa/N$, the choices for λ_t given in the Lemma satisfy the estimate (3.6) with $C_s = A_1\kappa^s$.

In view of this corollary, it suffices to assume that $\sqrt{1-t^2} \geq \kappa/N$, where κ can be chosen later at our discretion. We shall subsequently construct, for certain values of t, $\lambda_t = \lambda_{t,r,N} \in P_{N-2r+1}$ such that

(3.7) $\qquad |H_t - \lambda_t|(x) \leq C_r(\sqrt{1-t^2}/N)^{2r-1}|x-t|^{-2r+1}$ for $x \in [-1,1]$,

where r is any positive integer (to be chosen later depending on w and s). Furthermore, λ_t will be monotone, nondecreasing, with $\lambda_t(-1) = 0$ and $\lambda_t(+1) = 1$. Hence also

(3.8) $\qquad |H_t - \lambda_t|(x) \leq 1$ for all $x \in [-1,1]$.

Assuming these properties of λ_t for some t for the moment, we proceed to prove (3.6) for such t. Let $\delta = \sqrt{1-t^2}/N$. Then

$$\int_{-1}^{1} |H_t-\lambda_t||x-t|^{s-1}w(x)\,dx$$

(3.9)
$$\leq \int_{\substack{|x-t|\leq\delta\\|x|\leq 1}} |x-t|^{s-1}w(x)\,dx + C_r \int_{\substack{|x-t|\geq\delta\\|x|\leq 1}} |x-t|^{s-2r}w(x)\,dx\ \delta^{2r-1},$$

where we used the bounds (3.8) and (3.7) on the first and second integrals, respectively. Thus the estimate (3.6) follows easily from the following two assumptions:

(A2) Let s be a positive integer. Then there exist constants $A_2 < \infty$ and $\gamma \geq 2$ such that, for all $\delta > 0$ and $t \in [-1,1]$ satisfying $1-t^2 \geq \gamma\delta$,

$$\int_{|x|\leq\delta} w(x+t)|x|^{s-1}dx \leq A_2 w(t)\delta^s.$$

(A3) There exist $A_3 < \infty$, $\gamma \geq 2$ and $0 < k_0 < \infty$ such that, for all $\delta > 0$ and $t \in [-1,1]$ satisfying $1-t^2 \geq \gamma\delta$ and all $k \geq k_0$,

$$\int_{\substack{|x|\geq\delta\\|t+x|\leq 1}} w(x+t)|x|^{-k}dx \leq A_3 w(t)\delta^{-k+1}.$$

In applying (A2) and (A3) to (3.9), note that, for $\delta = \sqrt{1-t^2}/N$, the condition $1-t^2 \geq \gamma\delta$ is equivalent to $\sqrt{1-t^2} \geq \gamma/N$. This will therefore be satisfied if $\kappa \geq \gamma$, a requirement we now impose on κ. Choosing r such that $2r-s =: k \geq k_0$ thus proves (3.6) for t such that (3.7-8) hold. Before proving the bounds (3.7) and (3.8), we show that (A2) and (A3) hold for Jacobi weights.

Example 2: Assumption (A2) holds for the Jacobi weights $w(x) = (1+x)^\alpha(1-x)^\beta$ for all $\alpha,\beta \in \mathbb{R}$.

Proof. Take $\gamma = 3$. By a change of variables,

$$\delta^{-s}\int_{|x|\leq\delta} (w(x+t)/w(t))|x|^{s-1}dx$$

$$= \delta^{-s}\int_{|x|\leq\delta} \left(1 + \frac{x}{1+t}\right)^\alpha \left(1 - \frac{x}{1-t}\right)^\beta |x|^{s-1}dx$$

$$= \int_{|y|\leq 1} \left(1 + \left(\frac{\delta}{1+t}\right)y\right)^\alpha \left(1 - \left(\frac{\delta}{1-t}\right)y\right)^\beta |y|^{s-1}dy.$$

But for $1-t^2 > 3\delta$, $\delta/(1\pm t) \in [-\frac{2}{3}, \frac{2}{3}]$. Thus the integrand is bounded by $3^{\max\{|\alpha|,|\beta|\}}$. //

Example 3: Assumption (A3) holds for the Jacobi weights $w(x) = (1+x)^{\alpha}(1-x)^{\beta}$ for all $\alpha,\beta > -1$.

Proof. Take $\gamma = 2$ and $k_0 = 2 + \max\{\alpha,\beta,0\}$. Then

$$\delta^{k-1} \int_{\substack{|x|>\delta \\ |x+t|<1}} (w(x+t)/w(t))|x|^{-k}dx =$$

$$= \delta^{k-1}\left(\int_{\delta}^{1-t}(w(x+t)/w(t))x^{-k}dx + \int_{\delta}^{1+t}(w(t-x)/w(t))x^{-k}dx\right)$$

$$= \delta^{k-1}\left(\int_{\delta}^{1-t}(1 + \frac{x}{1+t})^{\alpha}(1 - \frac{x}{1-t})^{\beta}x^{-k}dx + \int_{\delta}^{1+t}(1 - \frac{x}{1+t})^{\alpha}(1 + \frac{x}{1-t})^{\beta}x^{-k}dx\right)$$

$$\equiv I_-(t,\delta,\alpha,\beta,k) + I_+(t,\delta,\alpha,\beta,k).$$

Since $I_+(t,\delta,\alpha,\beta,k) = I_-(-t,\delta,\beta,\alpha,k)$, it suffices to show that, for all $|t| < \sqrt{1-\gamma\delta}$,

$$I_-(t,\delta,\alpha,\beta,k) < C \text{ provided } k > 2 + \max\{\alpha,0\}.$$

By a change of variables, note that

$$I_-(t,\delta,\alpha,\beta,k) = \int_1^{(1-t)/\delta}(1 + (\frac{\delta}{1+t})y)^{\alpha}(1 - (\frac{\delta}{1-t})y)^{\beta}y^{-k}dy.$$

Let $M = (1-t)/\delta$. Thus, for $\alpha > 0$,

$$I_- < \int_1^M(1+y)^{\alpha}(1-M^{-1}y)^{\beta}y^{-k}dy$$

$$< 2^{\alpha}\int_1^M (1-M^{-1}y)^{\beta}y^{\alpha-k}dy$$

$$< 2^{\alpha}\int_1^M(1-M^{-1}y)^{\beta}y^{-2}dy.$$

In the first estimate above, we used the fact that, for $|t| < 1$,

$$(1+t) > \tfrac{1}{2}(1-t)(1+t) > \tfrac{1}{2}\gamma\delta = \delta.$$

For $\alpha < 0$, one has

$$I_- \leq \int_1^M (1-M^{-1}y)^\beta y^{-k} dy$$

$$\leq \int_1^M (1-M^{-1}y)^\beta y^{-2} dy.$$

A simple calculation shows that, for $M \geq 1$,

$$\int_1^M (1-M^{-1}y)^\beta y^{-2} dy \leq C_\beta \quad \text{for all } \beta > -1,$$

completing the proof. //

We now construct λ_t satisfying (3.7-8). We shall do so by choosing

(3.10) $$\lambda_t(x) := \int_{-1}^x \delta_t(y) dy,$$

where $\delta_t \in P_{N-2r}$ approximates the Dirac δ-function. In particular, δ_t will be nonnegative and have integral on $[-1,1]$ equal to one. Thus λ_t satisfies the claimed monotonicity property as well as having the required values at ± 1. Hence (3.8) will follow automatically.

To construct δ_t, first note that it suffices to take $\delta_t(x) := \frac{1}{2}$ for all $x \in [-1,1]$ for $N < 4r$ (recall that we restrict t to satisfy $\sqrt{1-t^2} \geq \gamma/N$ in (3.7-8)). For $N \geq 4r$, let n be the greatest integer not exceeding $N/2r$. Note that $N/n \leq 3r$. Let T_n denote the Chebyshev polynomial of degree n, and let $\{t_i : 1 \leq i \leq n\}$ be its zeros:

$$T_n(x) := \cos(n(\cos^{-1}x))$$

$$t_i := \cos((i - \tfrac{1}{2})\pi/n) =: \cos\theta_i; \quad i = 1,\cdots,n.$$

We shall establish (3.7-8) for $t \in \{t_i : 1 \leq i \leq n\}$. Define

(3.11) $$\delta_{t_i}(x) := c_i [T_n(x)/(x-t_i)]^{2r},$$

where c_i is chosen so that $\int_{-1}^1 \delta_{t_i}(x) dx = 1$. To estimate the size of c_i, observe that

$$|T_n(x)| = |\cos n\theta| \geq \tfrac{1}{2} \quad \text{for } |n\theta - i\pi| \leq \frac{\pi}{3}.$$

Thus, writing $x = \cos\theta$, we see that for $|n\theta - i\pi| \le \frac{\pi}{3}$

$$|T_n(x)/(x-t_i)| \ge \tfrac{1}{2}|\cos\theta - \cos\theta_i|^{-1}$$

$$\ge \tfrac{1}{2}\,|\cos(i+\tfrac{1}{3})\pi/n - \cos\theta_i|^{-1}$$

$$\ge [(5\pi/3n)\max\{\sin(i+\tfrac{1}{3})\pi/n, \sin\theta_i\}]^{-1}$$

$$\ge [(5\pi/n)\sin\theta_i]^{-1},$$

because $\sin\theta/\theta$ is decreasing on $[0,\pi]$. The measure of the set $\{x = \cos\theta : |n\theta - i\pi| \le \pi/3\}$ is

$$\cos((i-\tfrac{1}{3})\pi/n) - \cos((i+\tfrac{1}{3})\pi/n) \ge (2\pi/3n)\min\{\sin(i \pm \tfrac{1}{3})\pi/n\}$$

$$\ge (2\pi/9n)\sin\theta_i .$$

Therefore,

$$c_i^{-1} = \int_{-1}^{1} [T_n(x)/(x-t_i)]^{2r}dx$$

$$\ge ((5\pi/n)\sin\theta_i)^{-2r}(2\pi/9n)\sin\theta_i .$$

Thus

$$(3.12)\quad c_i \le 24(5\pi\sin\theta_i/n)^{2r-1} = 24(5\pi\sqrt{1-t_i^2}/n)^{2r-1} \le C_r(\sqrt{1-t_i^2}/N)^{2r-1}.$$

To complete the estimate (3.7), observe that

$$|H_{t_i} - \lambda_{t_i}|(x) = c_i \begin{cases} \int_{-1}^{x} \delta_{t_i}(y)dy & x \le t_i \\ \int_{x}^{1} \delta_{t_i}(y)dy & x > t_i . \end{cases}$$

Therefore,

$$|H_{t_i} - \lambda_{t_i}|(x) \le c_i \int_{|x-t_i|}^{2} y^{-2r}dy$$

$$\le c_i|x-t_i|^{-2r+1}/(2r-1).$$

Combined with (3.12), this proves (3.7) for $t = t_i$. Hence (3.6) is now verified for $t = t_i$.

For the general case $t_{i+1} < t < t_i$, define $\lambda_t := \lambda_{t_i}$. Note that, since $\sin\theta/\theta$ is decreasing on $[0,\pi]$,

$$\frac{1}{3}\,\pi\sqrt{1-t^2}/n < \pi \min\{\sin\theta_i,\sin\theta_{i+1}\}/n$$

$$< \int_{\theta_i}^{\theta_{i+1}} \sin\theta\, d\theta = t_i - t_{i+1}$$

$$< 3\pi \min\{\sin\theta_i,\sin\theta_{i+1}\}/n < 3\pi\sqrt{1-t^2}/n.$$

Therefore,

$$\int_{-1}^{1} |H_t-\lambda_t|(x)\,|x-t|^{s-1}w(x)\,dx$$

$$= \int_{-1}^{1} |H_t-\lambda_{t_i}|(x)\,|x-t|^{s-1}w(x)\,dx$$

$$< \int_{-1}^{1} |H_t-H_{t_i}|(x)\,|x-t|^{s-1}w(x)\,dx$$

$$+ \int_{-1}^{1} |H_{t_i}-\lambda_{t_i}|(x)\,|x-t|^{s-1}w(x)\,dx.$$

This first term, via (A2), is bounded by

$$\int_{t}^{t_i} |x-t|^{s-1}w(x)\,dx < A_2(t_i-t)^s w(t)$$

$$< A_2(t_i-t_{i+1})^s w(t)$$

$$< A_2(3\pi\sqrt{1-t^2}/n)^s w(t)$$

$$< C_{r,s}(\sqrt{1-t^2}/N)^s w(t).$$

(The application of assumption A2 is valid provided, e.g., $\min\{1-t_i^2, 1-t_{i+1}^2\} > \gamma(t_i - t_{i+1})$. The reader may easily check that this holds if the constant κ mentioned in the Corollary, and the subsequent discussion, is chosen sufficiently large, depending on r and γ. This observation applies as well to the application of assumptions A2 and A3 in the next set of inequalities.) Since (3.7-8) hold for t_i, the second term is bounded by $(\delta := \sqrt{1-t^2}/N)$

$$
\begin{aligned}
C_r \delta^{2r-1} & \int_{|x-t|>2|t_i-t_{i+1}|} |x-t_i|^{-2r+1} |x-t|^{s-1} w(x)\,dx \\
& + \int_{|x-t|<2|t_i-t_{i+1}|} |x-t|^{s-1} w(x)\,dx \\
& < C_r (2\delta)^{2r-1} \int_{|x-t|>2|t_i-t_{i+1}|} |x-t|^{s-2r} w(x)\,dx \\
& \quad + 2^s A_2 |t_i-t_{i+1}|^s w(t) \\
& < C_r (2\delta)^{2r-1} A_3 (2|t_i-t_{i+1}|)^{s-2r+1} w(t) \\
& \quad + 2^s A_2 |t_i-t_{i+1}|^s w(t) \\
& < \tilde{C}_{r,s} (\sqrt{1-t^2}/N)^s w(t),
\end{aligned}
$$

assuming (A2) and (A3) hold. Therefore, (3.6) is now proved for all $t \in [-1,1]$, and hence the estimate (3.1) is established for smooth u and for weights w satisfying assumptions (A1-3). In fact, in view of the Lemma, one can improve (3.6), and hence (3.1), by replacing the expression $(\sqrt{1-t^2}/N)^s$ by $\min\{(\sqrt{1-t^2}/N)^s, (1-t^2)^s\}$. To extend the result to more general u, define, for positive integers s,

$$
\|u\|_{w,s} = \sum_{k=0}^{s} \int_{-1}^{1} |u^{(k)}(x)|(1-x^2)^k w(x)\,dx, \tag{3.13}
$$

where $u^{(s)}$ is interpreted as a weak derivative. Define

$$
Y_w^s = \{u \in L^1_{loc}([-1,1]) : \|u\|_{w,s} < \infty\}. \tag{3.14}
$$

Then Y_w^s is a Banach space having $C^\infty([-1,1])$ as a dense subspace*. Using this density, we arrive at the following theorem, which summarizes our results.

Theorem 2. Let w be a positive, integrable function on $[-1,1]$ satisfying assumptions (A1-3), where s in (A2) is some positive integer. Let $u \in Y_w^s$ (see (3.13-4) for the definition). Then for any positive integer $N > s-1$,

$$\inf_{P\in \mathcal{P}_N} \int_{-1}^{1} |u-P|(x)w(x)\,dx < C_{s,w} \int_{-1}^{1} |u^{(s)}(x)| \min\{(\sqrt{1-x^2}/N)^s, (1-x^2)^s\} w(x)\,dx,$$

where $\mathcal{P}_N$ denotes polynomials of degree not exceeding N and $C_{s,w}$ is a constant independent of N and u.

Remark 1. If $u^{(s)}(x)(1-x^2)^{s/2}$ is integrable on $[-1,1]$, the above estimate may be simplified to yield

$$\inf_{P\in \mathcal{P}_N} \int_{-1}^{1} |u-P|(x)w(x)\,dx < C_{s,w} N^{-s} \int_{-1}^{1} |u^{(s)}(x)|(1-x^2)^{s/2} w(x)\,dx.$$

Remark 2. Assumptions (A2-3) imply, in particular, that $w(t) > 0$ for $|t| < 1$, unless $w \equiv 0$. But such a condition is necessary for an estimate such as (3.1) to hold. To see this, suppose that w is continuous at t and $w(t) = 0$. Choose a sequence $\{u_j\}$ of smooth functions with $u_j^{(s)}$ positive, supported in $\{t : |t| < 1/j\}$ and $\int_{-1}^{1} u_j^{(s)}(t)\,dt = 1$ (take $j > 1/(1-|t|)$ for simplicity). Then u_j converges in L^1 to $u(x) := H_t(x)(x-t)^{s-1}/(s-1)!$ as j tends to infinity. For $u = u_j$, the right hand side of(3.1) will tend to zero. However, the left hand side certainly will not do so since $u \notin \mathcal{P}_N$ for any N.

---

*Proof: Define $u_r(x) := u(rx)$, $0 < r < 1$. Then as $r \to 1$, $u_r \to u$ in Y_w. Let $u_{r,\varepsilon}$ be obtained from u_r by mollifying: $u_{r,\varepsilon} := u_r * \delta_\varepsilon$ where $\operatorname{supp}\{\delta_\varepsilon\} = [-\varepsilon,\varepsilon]$ and δ_ε has integral one and is smooth. Then $u_{r,\varepsilon}$ is smooth on $[-1,1]$ for $\varepsilon \le 1-r$, and $u_{r,\varepsilon} \to u_r$ in Y_w^s. Now let $r \to 1$, $\varepsilon := 1-r \to 0$ to get $u_{r,\varepsilon} \to u$ in Y_w^s.

4. Higher-order error estimates for Gaussian quadrature

In this section, we combine the results of previous sections to prove an estimate of the form (2.9). From (2.8), we know that

$$|e_N(f)| \leq C \inf_{P \in \mathbb{P}_{2N-2}} \int_{-1}^{1} |f'-P|(x)w(x)\,dx,$$

where $w(x) := \min\{\sqrt{1-x^2}/N, 1-x^2\}$. It is easy to check that, if a collection of weights w_i satisfy (A1), (A2) or (A3), then so does the weight function $\min\{w_i\}$. Thus Theorem 2 applies to the weight w since it is a minimum of two Jacobi weights, yielding

$$|e_N(f)| \leq C_s \int_{-1}^{1} |f^{(s)}(x)|\min\{(\sqrt{1-x^2}/N)^{s-1},(1-x^2)^{s-1}\}w(x)\,dx$$

$$= C_s \int_{-1}^{1} |f^{(s)}(x)|\min\{(\sqrt{1-x^2}/N)^{s},(1-x^2)^{s}\}\,dx.$$

We summarize this final result as

Theorem 3. *Let $e_N(f)$ denote the error in N-point Gaussian quadrature approximation to the integral of f on $[-1,1]$ (see (2.1-2) for definitions). Suppose that $(1-x^2)^s f^{(s)}(x)$ (weak derivative) is integrable on $[-1,1]$, i.e., $f \in Y_1^s$ where s is any integer such that $1 \leq s \leq 2N$. Then*

$$|e_N(f)| \leq C_s \int_{-1}^{1} |f^{(s)}(x)|\min\{(\sqrt{1-x^2}/N)^{s},(1-x^2)^{s}\}\,dx,$$

where C_s in independent of N and f.

Remark 3. If $f^{(s)}(x)(1-x^2)^{s/2}$ is integrable on $[-1,1]$, the above estimate simplifies to

$$|e_N(f)| \leq C_s N^{-s} \int_{-1}^{1} |f^{(s)}(x)|(1-x^2)^{s/2}\,dx.$$

This is the estimate anticipated in (1.1).

Acknowledgement. We wish to thank Professor G. P. Nevai for informing us of the work of Ky [1]. Professor Nevai has also developed an alternate (but related) method to that in section 3 for proving weighted approximation estimates.

REFERENCES

1. N. X. Ky, "On Jackson and Bernstein type approximation theorems in the case of approximation by algebraic polynomials in L_p-spaces," Studia Sci. Math. Hung. 9 (1974), 405-415.
2. G. Szegö, Orthogonal Polynomials, A.M.S. Colloquium Publications vol. XXIII, Providence, 4th ed., 1975.
3. A. F. Timan, Theory of Approximation of Functions of a Real Variable, Pergamon Press, Oxford, 1963.

RADV:LRS/db

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2269	2. GOVT ACCESSION NO. AD-A110478
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) ERROR ESTIMATES FOR GAUSSIAN QUADRATURE AND WEIGHTED-L^1 POLYNOMIAL APPROXIMATION	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ronald A. DeVore and L. Ridgway Scott	8. CONTRACT OR GRANT NUMBER(s) MCS-81-01661 DAAG29-80-C-0041 MCS-81-01631
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 3 - Numerical Analysis and Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18	12. REPORT DATE August 1981
	13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gaussian quadrature, polynomial approximation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Error estimates for Gaussian quadrature are given in terms of the number of quadrature points and smoothness properties of the function whose integral is being approximated. An intermediate step involves a weighted-L^1 polynomial approximation problem which is treated in a more general context than that specifically required to estimate the Gaussian quadrature error.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

DATE
FILMED
3-82

DTIC